

Three approaches to one-place addition and subtraction: Counting strategies, memorized facts, and thinking tools

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In many countries, the first significant chunk of elementary mathematics is the same: the numerals and addition and subtraction within 20. It has four pieces, which may be connected in different ways: the numerals from 1 to 20, computation of 1-place additions and related subtractions, additions and subtractions within 20 with 2-place numbers, and the introduction of the concept of addition and subtraction. This article will focus on the second piece — computation of 1-place additions and related subtractions, which I will call “1-place addition and subtraction” or “1-place calculation.”

Computational skills, in particular, the skills of mentally calculating 1-place additions and subtractions, are an important cornerstone for all four operations with whole numbers, decimals, and fractions in elementary school. Whether or not students are proficient in 1-place addition and subtraction will have a direct impact on their development of all later computational skills.

It seems easy to agree on the meaning of “1-place addition and subtraction”: addition and subtraction with sum or minuend between 2 and 18. Instructional approaches, however, vary. There are two main approaches that I have observed in US elementary schools: “counting counters” and “memorizing facts.” In this article, I will briefly describe these approaches. A third approach, which I call “extrapolation,” I shall describe in more detail.

In discussing instruction, we tend to notice two aspects: “what to teach” and “how to teach it.” The aspect of “how to teach” is two-fold—the level of curriculum and that of classroom teaching. I will focus on the curriculum level in describing the three instructional approaches to 1-place computation. Associated with each approach are different learning goals and assumptions about learning. These differences raise questions which may be useful for the field of elementary mathematics education to consider. At the end of this article, I raise three such questions.

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“Counting counters” and “memorizing facts”: Two approaches often used in the US

Counting counters

The “counting counters” approach focuses on helping students to get the answer for addition and subtraction by counting counters. Besides students’ own fingers, in early elementary US classrooms one can see various kinds of counters: chips of different shapes and colors, blocks of different sizes and materials, counters shaped like animals or other things that students like. Also, the dots on the number line posted on the classroom wall are often used as counters.

In the “counting counters” approach, the main task of instruction is to guide students to improve their counting strategies. As we know, computing by counting is the method children use before they go to school, it’s their own calculation method, and the counters most often used are their fingers. As research points out, the more experience children get with computation, the more they improve their strategies.¹ A concise summary of the strategies that children use is given in *Math Matters* (Chapin & Johnson, 2006, pp. 63 - 64). The process of optimizing addition strategies goes from *counting all*, to *counting on from first*, to *counting from larger*. For example, to calculate $2 + 5 = ?$ by counting all: show 2 fingers and show 5 fingers, then count all the fingers, beginning the count with 1. A more advanced strategy is counting on from first. Instead of counting from 1, a child calculating $2 + 5 = ?$ “begins the counting sequence at 2 and continues on for 5 counts.” In this way, students save time and effort by not counting the first addend. An even more advanced strategy is counting on from larger: a child computing $2 + 5 = ?$ “begins the counting sequence at 5 and continues on for 2 counts” In this way, students save even more time. Because we only have ten fingers, the strategy of counting all can only be used for addition within 10. But the other two strategies can be used to calculate all 1-place additions. There are also counting strategies for subtraction: “counting down from,” “counting down to,” “counting up from given” (Chapin & Johnson, 2006, p. 64).

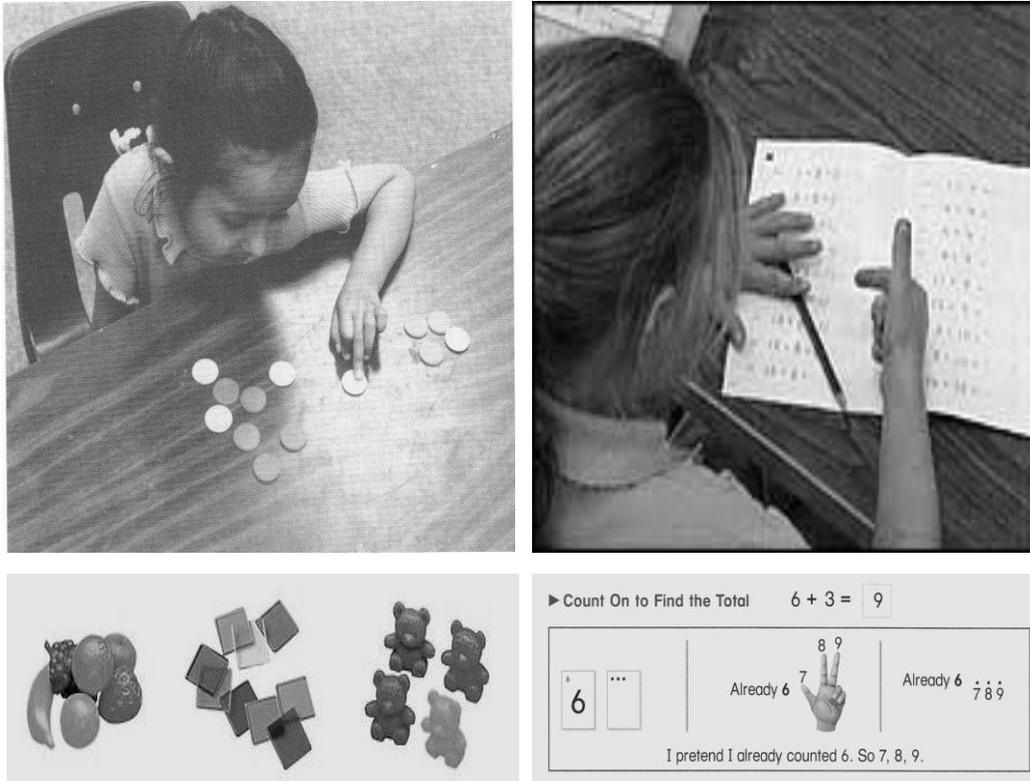


Figure 1. Student using counters, student using fingers, counters used in school, grade 2 textbook.

Counting strategies can be used to solve all 1-place additions and subtractions. However, the ability to use these strategies does not ensure that students develop the ability to mentally calculate 1-place additions and subtractions. The more proficient students become with counting strategies, the more they likely they are to develop the habit of relying on these methods. This habit may last for a long time. Thus, these methods may hinder the growth of the ability to calculate mentally. And, unless elementary mathematics education decides to give up the goal of teaching students the algorithms for the four operations, the ability to calculate mentally is necessary. Imagine that when students are doing multi-place addition and subtraction or multiplication and division, they still rely on counting fingers to accomplish each step of addition and subtraction. The whole process will become fragmented and lengthy, losing focus.

Memorizing facts

The goal of memorizing facts is to develop students' capability to calculate mentally. The content that students are intended to learn includes 200 "number facts." As we know, the ten digits of Arabic numbers can be paired in a hundred ways. The hundred pairs with their sums ($0 + 0 = 0$, $0 + 1 = 1$, , , , $9 + 9 = 18$) are called the "hundred addition facts." Corresponding to the "hundred addition facts" are "the hundred subtraction facts" ($0 - 0 = 0$, $1 - 0 = 1$, . . . , $0 = 9 - 9$). These two hundred facts, which cover all possible situations, are called "number facts." If one can memorize all these facts, then one is able to do mental calculations with 1-

place numbers. In this approach, the instructional method is to ask students to memorize the two hundred facts.

The number facts are shown to students with counters, then students are drilled. Drills are done in various ways. “Mad Minute” is an instructional practice often seen in US classrooms. A display shows 30 or 40 problems and students are asked to complete as many problems as they can in one minute. Each problem is written with the numbers in columns—similar to the form used in the algorithms for multi-place computations. Each sheet of drills may have a particular characteristic: all addition problems, all subtraction problems, or a mixture of the two. The numbers in the problems may lie within a particular range. For example, the left side of Figure 2 shows addition and subtraction problems within 10. There are also more interesting drill sheets (shown on upper right of Figure 2). A newer, more efficient variation of this method is to group related facts together, for example, grouping addition facts with the sum 7, by forming a “7-train” for students to memorize (shown on lower right of Figure 2).

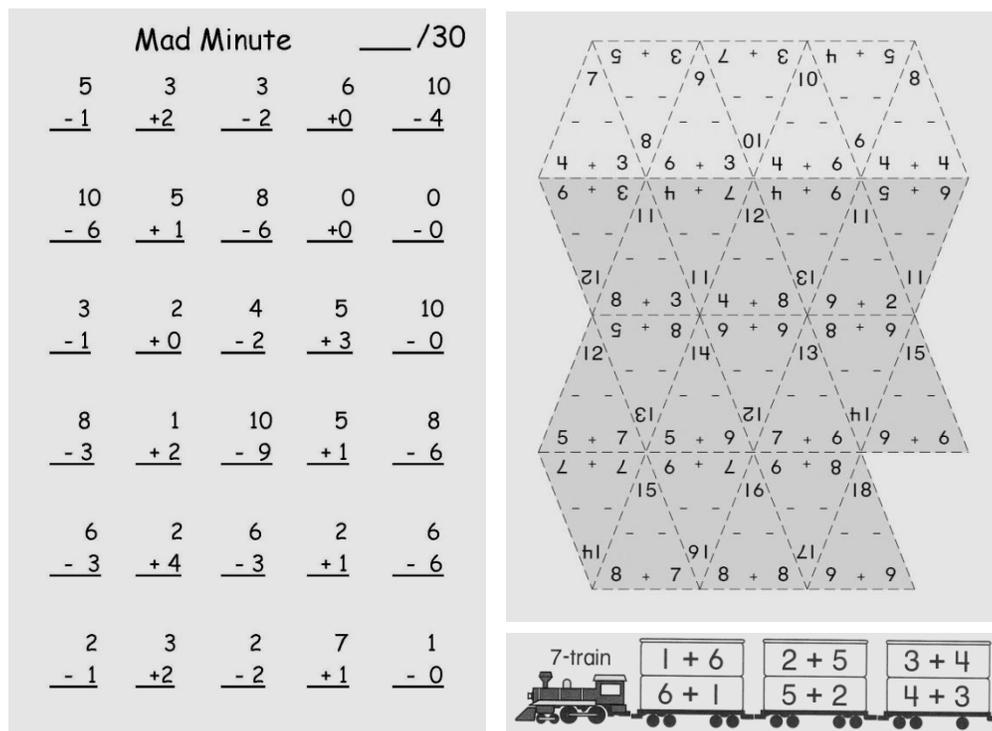


Figure 2.

As a way to develop students’ ability to do 1-place calculations mentally, the rationale for the memorizing facts approach is straightforward. However, its two elements, memorizing and considering 1-place calculation as 200 facts, have some defects with respect to activating students’ intellects.

First, although memorizing is an important intellectual function, children are considered to be especially good at memorizing. However, if students only memorize, then other types of

students' intellectual activities are neglected. Moreover, the task of memorizing items like number facts is more complicated than often thought. The memory of one item may interfere with the recall of another.ⁱⁱ

Second, when we consider 1-place calculation as 200 independent facts, the relationship and interdependence of these facts is neglected. Their differences in terms of difficulty and importance are also neglected. If they are 200 separate learning tasks, then there is no mutual support among the tasks. For example, if we memorize $4 + 3 = 7$ and $3 + 4 = 7$ separately, there is no support between these two learning tasks.

Recently, improvements have occurred in some US textbooks. They use the approach of grouping facts into “fact families,” for example, grouping $3 + 4 = 7$, $4 + 3 = 7$, $7 - 3 = 4$, $7 - 4 = 3$, hoping that the relationships shown among the facts can improve efficiency in memorizing. However, the relationships shown are limited. For example, in the 7-train in Figure 2, $1 + 6$ and $6 + 1$ are put in the top and bottom of the same car. The $6 + 1$ in the bottom car, which is easier for students, can support the task of learning the $1 + 6$ in the top car. However, can learning $6 + 1$ also support learning $7 - 6$ or $7 - 1$? Can it support $2 + 5$? Or $3 + 4$? Or $9 + 7$? Or $16 - 7$? If we consider memorizing facts as the goal of instruction, we will not make an effort to lead students to the deeper relationships that underlie the facts or to activate intellectual capacities other than memorizing.

Extrapolating: Using thinking tools to find unknown from known

The main idea of the counting approach is to allow students to use their hands—in particular, to use their fingers as counters and by counting to get the answer for a 1-place addition or subtraction calculation. The memorizing facts approach draws on memorization—through drill to memorize the 200 number facts. However, there is another approach to teaching 1-place computation—to let students “use their minds” to develop their capacity for mental computation. Because “using minds” involves inferring unknown from known, I call this approach “extrapolation.” It is used in China and some other countries around the world.ⁱⁱⁱ

Chinese elementary mathematics teachers often tell their students: “Use your little minds.” As they learn 1-place calculation, students are introduced to thinking tools. “Using minds” requires students to use these thinking tools to find an answer that they didn't know before. In other words, students don't depend on counters, in the form of fingers, objects, or textbook illustrations to get the answer. Instead, they use their own minds to figure out the answer.

The thinking tools that first graders use to extrapolate are mainly forms of “theoretical knowledge” in elementary mathematics, such as the basic knowledge of quantity (for example, the combination of 2 and 9), the knowledge of the relationship of quantities (subtraction as the inverse operation of addition), the knowledge of patterns in calculation (commutative law, compensation law, associative law, and basic understanding of notation, for example, base-10 notation for numbers less than 18 and the composition of 10). By using these thinking tools, first graders can increase their computational capabilities, step by step, and eventually master 1-place mental calculation.

**How “thinking tools” can help students to extrapolate:
The three kinds of calculation skills and the four stages of learning**

Extrapolation seems to be markedly more sophisticated than counting counters and memorizing facts. That may be due to the learning tasks involved. If the counting and memorizing facts approaches each have one learning task, then the extrapolation approach has two: to learn thinking tools while learning calculation skills.

1-place addition and subtraction involves three computational skills:

- A. addition and subtraction with sum or minuend between 2 and 9.
- B. addition and subtraction with sum or minuend of 10.
- C. 1-place addition and subtraction with sum or minuend between 11 and 18.

In the extrapolation approach these three skills are addressed in four stages. The first two stages of learning address skill A: calculation with sum or minuend between 2 and 9. The third stage addresses skill B and the fourth stage addresses skill C.

In terms of calculation skill, each stage includes new content. In terms of number range, each stage is cumulative, including the earlier numbers. For example, the skill addressed at Stage 2 is “addition and subtraction with sum or minuend between 6 and 9” but the number range is 1 to 9, including the numbers from the previous stage. Please see the following table.

Table 1. The three calculation skills and the four instructional stages

Three calculation skills	Four instructional stages	
	content	range of numbers
A. addition and subtraction with sum or minuend between 2 and 9.	1. addition and subtraction with numbers between 1 and 5.	1–5
	2. addition and subtraction with numbers between 6 and 9	1–9
B. addition and subtraction with sum or minuend of 10.	3. addition and subtraction with sum or minuend of 10.	1–10
C. 1-place addition and subtraction with sum or minuend between 11 and 18.	4. 1-place addition and subtraction with sum or minuend between 11 and 18.	1–20

The first of the four stages is based on the computational capacities that students bring to school. Each later stage is based on the capacities developed at prior stages. In the first part of this section, I discuss the kinds of thinking tools that are usually introduced in each of the four stages and the reasons for doing so. This discussion is based on my own experience and knowledge of teaching elementary mathematics in China, and and the analysis of seven Chinese textbooks published between 1988 and 2008. In the second part, I draw on examples from a Russian first grade textbook to show how these thinking tools can be introduced to young students.

Stage 1: Introducing the first thinking tools, drawing on the computational capacity that students already have

The first stage is “Calculations with numbers between 1 and 5.” During recent years, attention has been given to the idea that before they attend school, children already have developed these strategies, allowing them to do some simple calculations and has documented the systematic order in which these strategies develop. However, another important computational capability of children—their ability to mentally calculate small quantities—seems to not get enough attention from mathematics education researchers. *This ability is the foundation for the extrapolation method.*

Create a dialogue environment in which children are relaxed, happy, and willing to actively communicate with you, in their own words, about their everyday life, and make up word problems with quantities within 3, for example:

- If your dad gives two candies and your mom gives you one, how many did you get in all?
- You have three candies, I have one. Who has more? How many more?
- You had three candies and ate two. How many candies are left?

You will find that children will tell you the correct answer immediately and don't need to use fingers at all.

That children can fluently answer these questions shows they know the quantities, 1, 2, and 3 very well and already have:

1. the primary concept of addition and subtraction—being able to judge which operations to use in certain circumstances;
2. the skill of mental calculation within 3;
3. the concept of “quantity set”—are able to consider more than one unit (2 or 3) as one quantity;
4. the concept of the composition of number—have understood that 3 is composed by 2 and 1.

These understandings and capabilities are the seeds of their later mathematical thinking and also the footholds of the thinking tools used in the extrapolation approach.

The understanding of quantities within 3 and ability to calculate within 3 that children already have before attending school should come from the natural human ability to immediately perceive these quantities. The quantities 1, 2, and 3 are usually called perceivable quantities.

Almost everyone can recognize these without counting.^{iv} This ability can be observed in kindergarten students.^v

In fact, first grade teachers may even notice that the recognition of numbers 4 and 5, and computation within 4 and 5 are easy for first graders. In *Children's Counting and Concepts of Number*, there is an example of a child of 2 years and 10 months who commented, “Two and two make four” (Fuson, p. 19). That may have occurred because in everyday life children have many chances to become familiar with these quantities—every time they use their own hands, they will see the five fingers, four on one side and the thumb on the other. They have seen many times that 4 and 1 make 5, as well as the other combinations (for example, 2 and 2, 3 and 1, 3 and 2). For many people, 4 is a perceivable quantity as well as 3, 2, and 1.

In summary, calculation with numbers 1 through 5 seems to be already mastered by most students before they begin first grade. In this sense, “Calculations with numbers between 1 and 5” is a review.

However, this “review” still carries significant tasks. These tasks are to build new concepts, based on the knowledge of quantity and computational skill that students already have.

First, help students start to build the concept of abstract number, based on the concept of concrete number that they already have. Before they begin school, for most children “number” is “concrete number”—one person, two cards, three candies. What they compute are also concrete numbers. But, what is computed in mathematics are abstract numbers. Therefore, students need to be helped to make the transition from the concrete numbers that they are familiar with—3 people, 3 cards, 3 candies—to the abstract number 3.^{vi}

Second, to help students learn to use symbols to represent the quantities they already know, not only to build the concept of 3, but also to learn to use the numeral “3” to represent it. This is also an intellectual task.

Third, based on “3 is composed by 2 and 1” which students already know, further review “4 is composed by 2 and 2” and “4 is composed by 3 and 1,” “5 is composed by 3 and 2,” “5 is composed by 4 and 1” to help students build the concept “a number is composed of two smaller numbers” and further to notice the relationship between big and small numbers: because 3 is composed by 2 and 1, therefore $2 + 1 = 3$; because 5 is composed by 3 and 2, therefore $3 + 2 = 5$. Because students are familiar with the numbers from 1 to 5, this kind of understanding can be easily established. These understandings are thinking tools prepared for use at the next stage.

Fourth, help students to represent the calculations that they already know with equations. Based on what students already know—“if Dad gives me 2 candies and Mom gives me 1, then I will have 3 candies”—let them learn to express this calculation in the mathematical form “ $2 + 1 = 3$.”

Fifth, to help those students who haven't mastered addition and subtraction with sums of 4 or 5 to master this skill. Because of differences in family background or everyday environment, some students may not yet be comfortable in calculating these sums.

In summary, the task of teaching the first stage "Calculations with numbers between 1 and 5" is to take stock of the knowledge of quantities that students already have. Through this stocktaking, students develop a clearer understanding of this knowledge, at the same time establishing a conceptual foundation for their future mathematical learning. At this stage, there is no extrapolation. However, the learning tasks of this stage draw on students' prior knowledge in order to build a foundation for extrapolation in later stages.

Stage 2: Learn how to extrapolate with thinking tools

The second stage is "addition and subtraction with sum or minuend between 6 and 9." In contrast with the numbers from 1 to 5, the numbers 6, 7, 8, 9 are much less familiar to students. Thus, mental calculation with these numbers starts to feel harder to handle for students, although this is not uniform. Facing the calculations in this stage, many students unconsciously use their fingers to help. At this moment, instruction comes to a fork in the road: letting students acquire the method of counting on fingers or leaving them to use a new method: using mathematical thinking tools to extrapolate from known to unknown. In other words, what aid should students use: fingers as counters or mathematical thinking tools? Extrapolation chooses the latter. The five teaching tasks accomplished at the previous stage have prepared students to learn extrapolation at this stage.

The difficulty at this stage is that students are not familiar with the four numbers, 6, 7, 8, 9. To dissipate this difficulty, the prior stage prepared students in two ways. First, based on students' prior knowledge that "2 is composed by 1 and 1" and "3 is composed by 2 and 1," by revealing that "4 is composed by 3 and 1," "4 is composed 2 and 2," and "5 is composed by 3 and 2," the concept of "a big number is composed of smaller numbers" was established. Second, from the tasks they did at the previous stage, students became more familiar with the numbers from 1 to 5. Thus, although students are not familiar with 6, 7, 8, 9, they have established the concept of *a large number is composed of smaller numbers*, and they are very familiar with the smaller numbers that compose 6, 7, 8, 9. They have a foundation for understanding these larger numbers, lessening the initial difficulty. For example, a student might feel that 6 is unfamiliar and hard to comprehend. However, when he or she sees that 6 is composed of 5 and 1, of 4 and 2, and of 3 and 3, because the numbers from 1 to 5 are already familiar, 6 becomes less unfamiliar. The student is one step closer to comprehension of 6.

Also, students already learned that because 3 is composed of 2 and 1, therefore $2 + 1 = 3$. Thus, when they see 6 is composed of 5 and 1, 6 is composed of 4 and 2, 6 is composed of 3 and 3, without counting, they can extrapolate that $5 + 1 = 6$, $4 + 2 = 6$, $3 + 3 = 6$. Extrapolation based on the *composition of numbers* relies on the comprehension of previous numbers. At this stage, each new larger number is treated separately, in increasing order: sums and minuends of 6, then of 7, 8, and 9.

The commutative law of addition is also a useful thinking tool for extrapolation. Generally, adding a small number to a large number is easier than adding a large number to a small number. For example, a student who doesn't feel that it's hard to calculate $5 + 2$ may feel it's hard to calculate $2 + 5$. However, if the student knows the commutative law of addition, he or she will be able to extrapolate $2 + 5 = 7$ from $5 + 2 = 7$ which was already known. Another example: adding 1 is not difficult for students at all. Therefore, $5 + 1$, $6 + 1$, $7 + 1$, $8 + 1$, $9 + 1$ will feel very easy. But, calculating $1 + 5$, $1 + 6$, $1 + 7$, $1 + 8$ is not that easy. By knowing the commutative law of addition, students can draw on easy calculations to do harder ones.

Also, we know that subtraction is harder than addition. For example, $9 - 2$ is harder than $7 + 2$. However, if students understand that *subtraction is the inverse operation of addition*, that subtraction is to find an unknown addend, then they can extrapolate the result of subtraction from knowledge of a related addition—for example, extrapolate $9 - 2 = 7$ from $7 + 2 = 9$. In China and some other countries, addition and subtraction are usually taught at the same time. Chinese teachers usually remind students “When doing subtraction, think about addition.” This encourages students to support the learning of subtraction by using computational skills for addition that they have already mastered or find easier to use.

Leading students to observe the *compensation law of addition* and its *application to subtraction* can also promote students' ability to extrapolate. For example, $6 + 1 = 7$ is easy for students. Based on compensation law of addition, from $6 + 1 = 7$, students can extrapolate $5 + 2 = 7$, $4 + 3 = 7$, also extrapolate $6 + 2 = 8$, $6 + 3 = 9$, . . . Based on the application of the compensation law of subtraction, from $8 - 1 = 7$, students can extrapolate the solutions of $8 - 2 = ?$, $8 - 3 = ?$, $8 - 4 = ?$, $8 - 5 = ?$, and so on.

In summary, the composition of a number, the commutative law of addition, subtraction as the inverse operation of addition, and the compensation laws of addition and subtraction are all thinking tools that students can use when they encounter obstacles at this stage. Several of these thinking tools can be used together so that students know how to use different approaches to solve the same problem. Students can also choose a favorite tool or approach to solve a problem. In brief, with the help of thinking tools, at this stage it is possible for students calculate mentally without counters.

Compared with the counting approach which also depends on students finding the solution of addition and subtraction problems, the “disadvantage” of the extrapolation approach is that it requires more time and intellectual effort. The counting approach draws on strategies that students already know, therefore, the time needed for instruction is shorter (mainly to help some students to replace lower-level strategies with more advanced ones, e.g., replacing counting all with counting from first). The extrapolation approach needs not only preparation at the previous stages, but also requires time to introduce new thinking tools at each stage.

However, the extrapolation approach has advantages that the counting approach does not have:

- Learning is more robust. Whether in terms of understanding the numbers 6, 7, 8, 9, or addition and subtraction with any of these numbers as sum or minuend, after students have actively participated in computation with different strategies, they get a deep

impression and develop a solid mastery, finding these sums and differences easy to learn by heart. In contrast, students using counting strategies are less likely to learn these sums and differences by heart.

- Once students finish the mental calculation tasks at this stage, they stop having the impulse to count their fingers when calculating. They have not only learned additions and subtractions with numbers within a certain range, but more importantly, have started to develop particular approaches for mental calculation, initiated the habit of mental calculation, and have gained confidence with it.
- At the beginning of first grade, students get exposed to and practice the method of using thinking tools to extrapolate an unknown solution from prior knowledge. This perspective—using thinking tools to find the unknown from the unknown—initiates a lasting attitude toward learning and mathematics.

Due to these three features, the learning at this stage becomes a foundation for future learning—the foundation for the next stage and successive stages as well as the foundation for all of elementary mathematics learning.

Stage 3: Numeral system and the composition of ten

Because of the special role that 10 plays in the base-10 system, this number is important enough to have a separate learning stage. This number is important conceptually and with respect to calculation.

Conceptually, 10 is the first 2-place number that students learn. It is also the first time they are exposed to positional notation, where a numeral's position indicates the quantity that it represents.

In terms of computational skills, the composition of 10 from pairs of numbers plays a crucial role in students' mastery of addition with composing and subtraction with decomposing.

The five compositions of 10 (9 and 1, 8 and 2, 7 and 3, 6 and 4, 5 and 5) are hard to get into students' heads via rote learning. Two things will help students extrapolate at this stage. Comprehension of the numbers from 1 to 9 and the thinking tools, both of which were acquired at the two previous stages. Making a ten will become a thinking tool used for extrapolation at the next stage.

Stage 4: Deepening the knowledge of numeral system and introducing associative law of addition

“1-place addition and subtraction with sum or minuend between 11 and 18,” also known as “addition with carrying and subtraction with borrowing within 20,” is a foundation needed for addition and subtraction with whole numbers and decimals. If the learning task of this stage is not well accomplished, students will not be able to master the algorithms for the four operations. However, for first graders the calculations associated with this stage are hardest.

On the one hand, the numbers involved are large, so large that they are beyond what students can comprehend. On the other hand, the notation involved is positional notation, involving the abstract concept that the place of a numeral changes its value. Indeed, the difficulty of this stage is an intellectual height that most first graders are not able to scale in one attempt. However, at the previous three stages, instruction led students to use extrapolation to climb, step by step, dissipating the difficulties of these stages.

At this stage, along with the knowledge gained at the first three stages, extrapolation requires two more thinking tools: understanding of the *meaning of the numerals from 11 to 20*, based on positional notation, and the *associative law*.

With these tools, students can solve the problems of this stage, using their minds, without fingers or other counters. For example, based on the knowledge that students already have, they can tell that the answer for $9 + 2 = ?$ will be larger than 10. But, what is the exact number? By understanding the associative law and by knowing the composition of 9 and 2, they can separate a 1, which with 9 composes a ten, from 2; make a ten, with 1 remaining. Based on positional notation, one ten and one one is written as 11. Therefore, $9 + 2 = 11$. From this, by using the commutative law and subtraction as the inverse operation of addition, students can extrapolate $2 + 9 = 11$, $11 - 2 = 9$, $11 - 9 = 2$. Using the same idea, they can extrapolate solutions for all the problems of this stage.

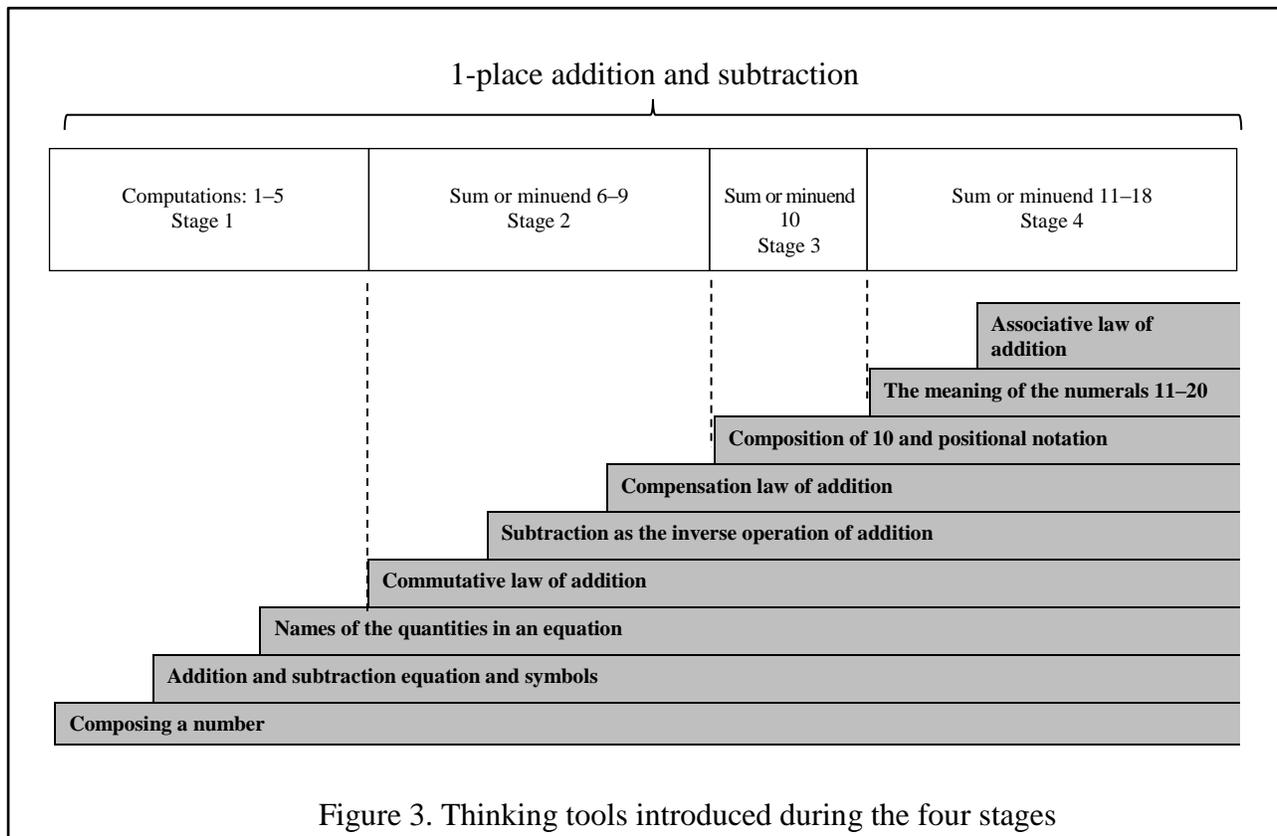
Moreover, once they use the associative law to find a solution, students can also use the compensation law to extrapolate further additions and subtractions. For example, from $9 + 2 = 11$, they can extrapolate $9 + 3 = 12$, $9 + 4 = 13$, $9 + 5 = 14$, $9 + 6 = 17$, and so on. Each of these equations, with extrapolation, generates another group of equations. The thinking tools and computational skills that students have already gained prepare them to use several ways to extrapolate.

When the learning of this stage is accomplished, first graders' capacity for mental calculation with 1-place numbers has been achieved. This capacity will contribute to their learning to use algorithms for the four operations with multi-place numbers. Of course, future calculations will reinforce this capacity.

Overview of the thinking tools

The extrapolation approach just described can help students develop the capacity to calculate mentally with 1-place numbers, without counters, as used in China and some other countries. However, this cannot be done instantly, it requires sustained effort from students and teachers for about 20 weeks.^{vii}

Figure 3 shows how the various thinking tools are introduced over four stages. The dotted lines separate the four stages.



In Figure 3, we see that in the process of developing the capacity of mental computation the extrapolation approach leads students to climb nine steps. The instructional content of these steps is thinking tools needed for extrapolation or preparation for the introduction of a thinking tool. Except at the first stage, the thinking tools introduced at a given stage are used at that stage for the learning tasks of that stage and all later stages. Moreover, these thinking tools are useful for not only 1-place addition and subtraction, but will be used throughout elementary mathematics.

**How to introduce “thinking tools” to first graders:
Examples of being intellectually honest and learner-considerate at same time**

The next problem is: How can thinking tools such as the commutative law, the idea of inverse operation, and the associative law be introduced to students at the beginning of first grade? In this section, we will illustrate how this can be done with examples from the first grade Russian textbook translated by the University of Chicago School Mathematics Project (Moro et al., 1980/1992). In aspects such as when to and how to introduce the thinking tools, the Russian arrangement differs slightly from the Chinese, but the basic idea of the extrapolation approach is the same.

The lessons in the Russian textbook can be grouped in two main categories: Introducing concepts (along with exercises), exercises only. The examples of lessons that follow, which introduce thinking tools, all belong to the first category. These lessons have three sections: title, introduction of new concept, related exercise problems.

The titles of lessons that introduce thinking tools are usually not the names of those thinking tools but the key words describing the learning task of the lesson. For example, the lesson that introduces the commutative law of addition is called “Interchanging Addends” and the lesson that introduces subtraction as the inverse operation of addition is called “How to Find an Unknown Addend.”

The section that introduces the new concept usually combines one or more pictures and equations. The pictures have themes familiar to students, accompanied by appropriate equations. The quantities in the pictures and equations are usually small numbers that students already know which have been addressed in previous stages.

The exercise sections are usually composed of several groups of exercises that increase in difficulty within each group and from group to group. The problems have different forms, such as equations, pictorial problems, and word problems. To understand a concept, one needs to use it: The exercises allow students to apply the new ideas that they have just learned and deepen their understanding.

The first two of the following six examples are initial segments of lessons. The remaining examples are entire lessons. (The example of the compensation law is not from the Russian textbook but data collected from a Chinese first grade classroom.)

Addition and subtraction equations and associated symbols

Before they enter school, students can perceive quantities and have preliminary concepts of addition and subtraction. Introducing the mathematical representations of these concepts is the first step in learning mathematics. The following lesson introduces addition and subtraction equations. In earlier lessons, students have learned to use the Arabic numerals 1, 2, 3.

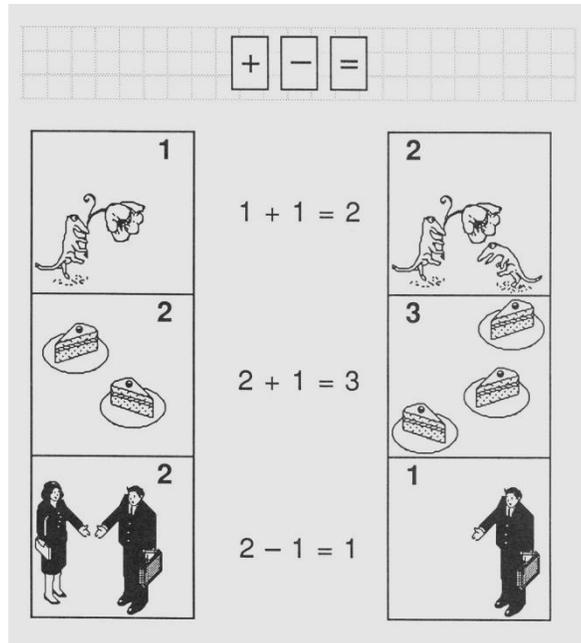


Figure 4. (Moro et al., 1980/1992, p. 9)

The title of the lesson is “+, −, =”—the symbols needed to connect numbers to make addition and subtraction equations. The section for introducing the new concept is composed of three pairs of pictures, each pair accompanied by an equation. The pictures show objects that are familiar to students. The addition and subtraction equations “ $1 + 1 = 2$,” “ $2 + 1 = 3$,” “ $2 - 1 = 1$ ” look very simple. However, they are expressed in the standard notation used throughout the world, which is the result of centuries of notational evolution.^{viii} By using the notation and form shown in these equations, students can express their own mathematical knowledge in a way that allows them to communicate with the rest of the world.

What students learn in this lesson is only the format for mathematical expressions. There seems to be no new conceptual content. However, students still make significant intellectual progress. To go from the everyday “two pieces of cake and one piece of cake make three pieces of cake” to the mathematical equation “ $2 + 1 = 3$ ” involves two intellectual leaps. One is from concrete numbers to abstract number. The other is from everyday language to the mathematical language used by the rest of the world.

Another meaningful arrangement in the lesson that students might not notice: when they first encounter addition and subtraction equations, the two operations occur in the same lesson. In the previous lessons on addition and subtraction, these concepts occurred independently. In this lesson, a connection between the two operations is unobtrusively represented. It is a preparation for later revealing the common quantitative relationship that underlies addition and subtraction.

The names of quantities in addition equations

After students get familiar with addition and subtraction equations, they are introduced to the names of the quantities in these equations. Figure 5 shows the introduction of new concept section of the lesson that introduces the names of quantities in addition equations. (The lesson title is part of this section as shown in the figure.)

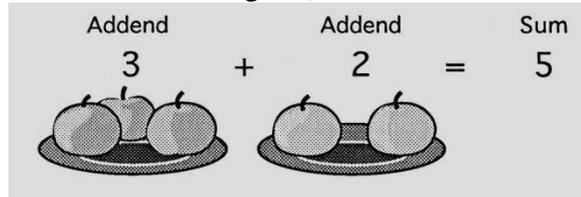


Figure 5. (Moro et al., 1980/1992, p. 38)

A combination of pictures and equations and text introduces the names of the two different types of quantities in an addition equation—addend and sum—helping students to abstract the general notion of “addend plus addend equals sum” from their understanding of specific addition equations: $3 + 2 = 5$, $2 + 2 = 4$, $2 + 1 = 3$, . . .

Introducing students to the names of the quantities in addition equations serves two goals, one short-term and one long-term. The long-term goal is to lead students to comprehend the quantitative relationship that underlies all four operations. The short-term goal is to prepare students for the introduction of the thinking tools for 1-place addition and subtraction.

Commutative law of addition

The commutative law can be used as the first thinking tool for students to learn extrapolation. The following lesson (Moro et al., 1980/1992, p. 51) introduces the commutative law at the stage of “addition and subtraction with sum and minuend from 1 to 9.”

Interchanging Addends.

$2 + 1 = 3$
 $1 + 2 = 3$

 1. $3 + 1$ $1 + 3$	 $4 + 2$ $2 + 4$	 $5 + 1$ $1 + 5$
------------------------------	------------------------	------------------------

2. $1 + 4 = 5$	$3 + 4 = 7$	$3 + 2 = 5$
$4 + \square = 5$	$4 + \square = 7$	$2 + \square = 5$

A sum does not change if the addends are interchanged.

Figure 6. (Moro et al., 1980/1992, p. 51)

The title of the lesson is “Interchanging Addends.”

The section that introduces the new concept depicts an everyday scene that students might encounter, accompanied by two equations $2 + 1 = 3$ and $1 + 2 = 3$. In this section, students don't need to calculate. Their learning task is to observe and discuss the new concept, guided by the teacher.

The exercise section includes two groups of problems related to the concept just introduced. The first group of problems leads students to notice that the phenomenon illustrated by the top picture also applies to other numbers: when addends are interchanged, the sum does not change. From the pictures, students can clearly see why there is such a pattern. The second group of exercises are calculations. Students are supposed to extrapolate the solution for each lower equation from the one above it, based on the pattern that they just noticed. This is also a chance for students to check the validity of the pattern they observed.

Eventually, after all these intellectual activities, the lesson presents the statement of the commutative law of addition. By learning the commutative law, students acquire a thinking tool that can be used for extrapolation. This tool will be used mainly to extrapolate the sum of a small number and a large number from the sum of the large number and the small number. On later pages of the textbook (pp. 52, 53, 54), many groups of exercises involve finding the sum of a small number and a large number, allowing students to use the commutative law to extrapolate the sum, and appreciate the convenience of this strategy.

In the textbook, the thinking tools for calculation at later stages are usually introduced by pictorial problems involving numbers from previous stages, with which students are already familiar. For example, although the lesson above occurs at the second stage, the section that introduces the new concept uses numbers from the first stage. In the exercise section, students can use the new concept to solve problems with numbers at the second stage. This approach has several advantages:

1. the smaller numbers reduce the cognitive load of calculation, allowing students to focus on the new concept;
2. effective use of the knowledge that students already have to support the acquisition of new knowledge;
3. utilizing knowledge that students already have allows them to see new features of that knowledge that deepen their understanding;
4. the new thinking tools students just learned are immediately used for calculations with larger numbers, helping students to solve new kinds of problems.

Students can appreciate the significance of these tools and also get a chance to use them.

There is one more interesting aspect that deserves mention. Although the task of the lesson is to introduce the commutative law of addition from its title “Interchanging Addends” to the end, in the statement “A sum does not change if the addends are interchanged” the term “commutative law of addition” never appears. We can see that the lesson is designed to be intellectually honest but also considerate of learners by not containing anything superfluous. These principles, to efficiently utilize the knowledge that students already have, and to be intellectually honest and at the same time considerate of learners, occur in every lesson that introduces a new concept.

Subtraction as the inverse operation of addition

Subtraction as the inverse operation of addition is another thinking tool used at the stage of “addition and subtraction with sums and minuends between 6 and 9.” This concept is introduced in the lesson entitled “How to Find an Unknown Addend.”

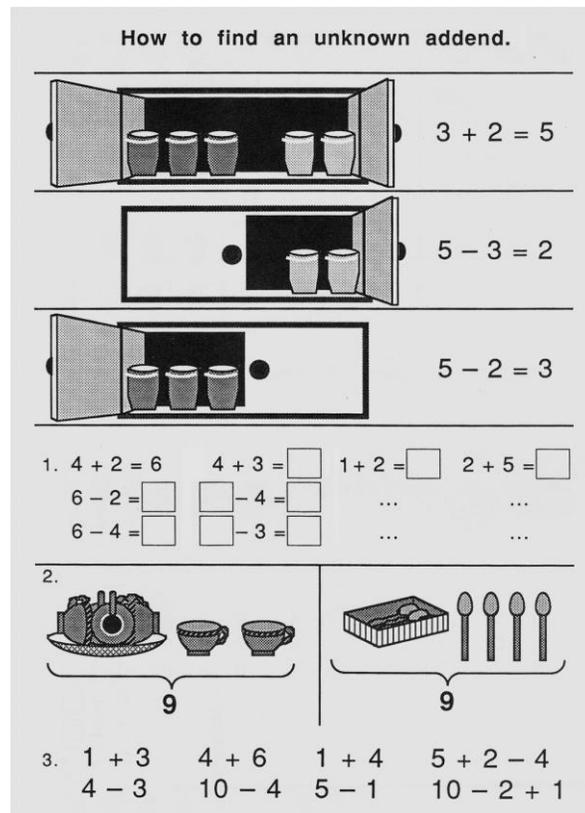


Figure 7. (Moro et al., 1980/1992, p. 55)

The quantities used in the section that introduces the new concept are the numbers from the first stage. The three pictures are like three cartoon panels. At the top are five jars in a cupboard whose doors are open: three at left and two on the right. This picture illustrates the equation “ $3 + 2 = 5$.” In the next picture, the lefthand cupboard door is closed, hiding the three jars. Only two of the five jars can be seen. This picture illustrates the equation “ $5 - 3 = 2$.” The bottom picture shows the left door open and the right door closed, hiding two jars and illustrating “ $5 - 2 = 3$.”

The quantity of five jars and the arrangement of the jars in the same in all three pictures. The only change in the pictures is the closing of different doors. The equation, however, changes from an addition equation to two subtraction equations. The changes in the three pictures and their accompanying equations reveal for students the relationship between addition and subtraction—addition is to find the sum of two known addends and subtraction is to find an unknown addend when the sum and one addend are known.

Because subtraction, the inverse operation of addition, is to find an unknown addend, the result of subtraction can be extrapolated from knowledge of addition, which is easier to master. When seeing a subtraction, and not knowing the solution, one only needs to think “What number when added to the subtrahend yields the difference?” As illustrated in the lesson, if one knows “ $3 + 2 = 5$,” then one can extrapolate: $5 - 3$ certainly must be 2 and $5 - 2$ certainly must be 3.

The three groups of problems in the exercise section are to help students deepen their understanding of the concept and at the same time learn to use the thinking tool just learned. The first group of problems has four subgroups. The first subgroup presents an additional equation with sum: $4 + 2 = 6$. From this equation, with the concept of subtraction as the inverse operation of addition just learned, students can extrapolate the answer of the two problems under it: $6 - 2 = ?$ and $6 - 4 = ?$ The second subgroup presents an addition. Students are supposed to find the sum. Based on the addition equation, students can fill in the boxes to create subtraction equations. The third subgroup presents an addition with small numbers “ $1 + 2$.” Students are supposed to find the answer, then create two subtraction equations on their own. The fourth subgroup presents a more difficult addition “ $2 + 5$.” Students again create two subtraction equations. The difficulty of the problems increases from subgroup to subgroup, but each subgroup prepares students for the next one by deepening their understanding of the relationship between subtraction and addition.

The second group of problems are pictorial problems about finding an unknown addend. Both problems involve two different types of objects, illustrating the sum of two numbers. The sum is represented as a number. One addend is clearly represented and the other is not. Each problem involves the same sum, but a different known addend. The problem on the left has two cups and the problem on the right has four spoons. As with the word problems, to solve a pictorial problem, students are supposed to first compose an equation corresponding to the problem, and then find the solution of the equation. The content of this lesson allows students to practice how to compose a subtraction equation to find the addends that are not clearly represented in the pictures.

The third group of exercises is composed of four subgroups of computations. The first three subgroups allow students to use the concept of inverse operation to find the solutions. The last subgroup is multi-step operations which prepares students to learn the associative law.

After this lesson, there are six groups of problems, making 36 problems in all, allowing students to practice using extrapolation to find a subtraction from a known addition.

Again, although the concept introduced in the lesson is subtraction as the inverse operation of addition, the term “inverse operation” did not occur in the lesson. In the title of the lesson, “How to find an unknown addend” has only one unfamiliar term “unknown.” This wording seems also to reflect the principle of being considerate of learners by not containing anything superfluous.

Compensation law

The two basic quantitative relationships in elementary mathematics, the sum of two numbers and the product of two numbers both involve three quantities. Any three quantities that are related show the following pattern: if one quantity remains unchanged, the change in the other quantity will be related. For example, the sum 2 and 3 is 5. If the sum 5 remains unchanged, then if the first addend 3 increases, then the second addend 2 must decrease by the same amount. Otherwise, the quantities do not maintain the same relationship. This is the law of compensation. There are corresponding compensation laws for subtraction and division, the inverse operations for addition and multiplication. Many computations can be made easier by use of the compensation laws.^x

In the Russian first grade textbook, I did not find examples of how to introduce the compensation law to young students. In a Chinese first grade classroom, I observed a teacher leading her students “to find the patterns”—the patterns of the change of quantities in addition and subtraction equations. The teacher came to the classroom with three small blackboards, each with a group of five equations. The bottom two equations of each group were covered by a piece of paper so that students couldn’t see them. During the lesson, the teacher took out the first small board and led students to “find a pattern” among the three top equations. After an active observation and discussion students noticed the pattern: in these equations, going from top to bottom, the first addend decreases by 1 every time, the second addend increases by 1 every time, the sum is unchanged. Going from bottom to top, the first addend increases by 1 every time, the second addend decreases by 1 every time, the sum is still unchanged. They also found that between first and third equation, the change range is 2, but the sum is also unchanged. Then the teacher removed the covering paper and students saw the last two equations, each with a blank box. They immediately figured out what numbers should go in these boxes. In the same manner, the class examined and discussed the other two small blackboards and learned the other two patterns: “One addend increases, the sum increases correspondingly” and “Subtrahend increases, difference decreases correspondingly.”

$5 + 1 = 6$	$4 + 1 = 5$	$6 - 1 = 5$
$4 + 2 = 6$	$4 + 2 = 6$	$6 - 2 = 4$
$3 + 3 = 6$	$4 + 3 = 7$	$6 - 3 = 3$
$2 + \square = 6$	$4 + 4 = \square$	$6 - 4 = \square$
$1 + \square = 6$	$4 + 5 = \square$	$6 - 5 = \square$

Figure 8. Leading students to notice the compensation law

In all of the examples above, students can observe which quantity is unchanged, which quantities change, and the pattern of change. Once they find the pattern of change, they can fill in the blanks and use the pattern in calculations with other numbers.

Associative law and single-place addition with sums from 11 to 18

The associative law is very important for helping students to understand addition with composing and subtraction with decomposing. It is also an important thinking tool for extrapolation at this stage. The textbook uses four lessons to introduce the associative law for addition and its application to subtraction: “Adding a Number to a Sum” (p. 106), “Subtracting a Number from a Sum” (p. 113), “Adding a Sum to a Number” (p. 125), “Subtracting a Sum from a Number” (p. 142).^{xi} This arrangement takes care of different ways in which the associative law is used, beginning with topics that are easy for students and progressing to more difficult ones. The structure of these four lessons is similar: a section that introduces the new concept, composed of three lines of cartoon, each with three panels accompanied by the equations they illustrate. The cartoon topic for addition is birds on a tree and the topic for subtraction is fish in a tank. The exercises of the lesson are groups of calculations that ask students to solve each problem in three different ways. Because adding a sum to a number is key to helping students understand the rationale for addition with composing, the lesson on “Adding a Sum to a Number” will be used as an example.

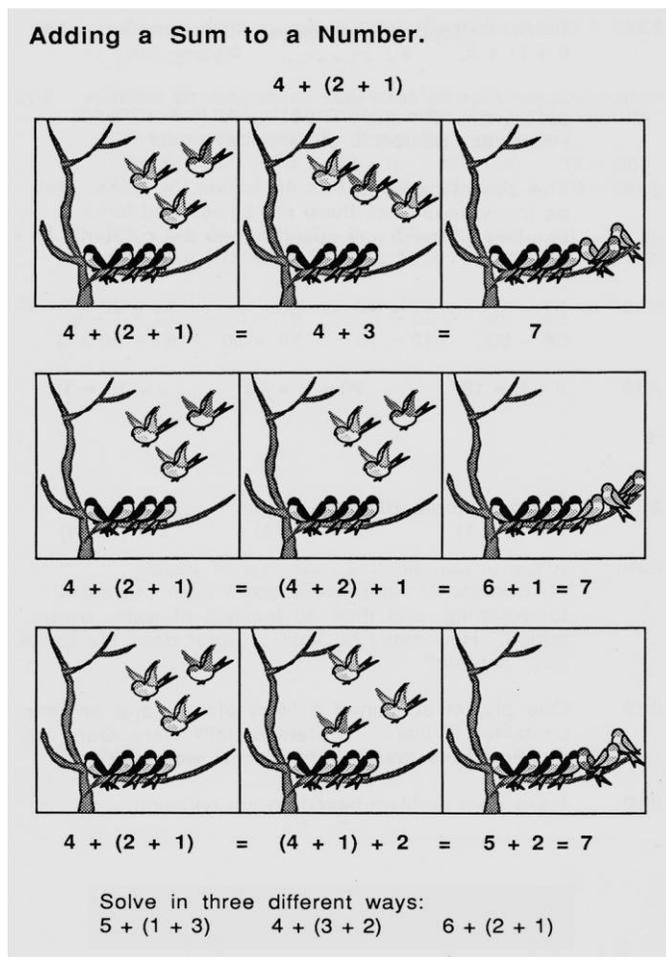


Figure 9. (Moro et al., 1980/1992, p. 125)

The title of the lesson, “Adding a Sum to a Number,” is represented by the expression $4 + (2 + 1)$ —adding the sum of 2 and 1 to 4. Again, we see that when a thinking tool is introduced, the numbers used are those of an earlier stage, already mastered by students. The pictures accompanied by small numbers will allow students to focus on the concept without a cognitive load induced from calculation.

The three lines of cartoons seem to tell three different stories. The beginning and end of the three stories are the same, but the second panels are all different. The beginning shows 4 birds on a tree branch and 3 birds flying, 2 in front and 1 behind. This illustrates the expression $4 + (2 + 1)$.

The second panel of the first story still shows 3 birds flying, but they are now all in one line, and seem to be arriving at the branch at the same time to join the 4 birds. The expression under the panel is $4 + 3$.

The second panel of the second story shows the 3 birds in the same configuration as at the beginning, 2 in front, 1 behind. It seems as if the first 2 birds will join the 4 on the branch sooner than the last bird. It illustrates $(4 + 2) + 1$.

The second panel of the third story shows the 3 flying in a different configuration, 1 is front and 2 are behind. It seems as if the first bird will join the 4 on the branch sooner than the 2 birds behind. The panel illustrates $(4 + 1) + 2$.

The three stories end in the same way, 7 birds sit on the branch. These three stories reveal that starting from the same expression, $4 + (2 + 1)$, one may go through three different computational processes, but end with the same result.

$$4 + (2 + 1) = 4 + 3 = 7$$

$$4 + (2 + 1) = (4 + 2) + 1 = 6 + 1 = 7$$

$$4 + (2 + 1) = (4 + 1) + 2 = 5 + 2 = 7$$

The section of exercises is a group of problems that involve adding a sum to a number. Students are supposed to solve with each with three different approaches. The numbers used are from the second stage, no composing is involved.

A sum can be added to a number in different ways: the entire sum can be added to the number at once, or the sum can be added addend by addend. Learning these ways to add a sum to a number prepares students to learn 1-place addition with sums from 11 to 18 as illustrated in the following lesson.

$9 + 5$

$9 + 5 = 9 + (1 + 4) = (9 + 1) + 4 = 14$
 $8 + 3 = 8 + (2 + 1) = (8 + 2) + 1 = 11$

Solve, and explain your answers:
 $9 + 7$ $8 + 5$ $7 + 6$ $6 + 5$

Figure 10. “1-place addition with sums from 11–18” (Moro et al., 1980/1992, p. 138)

Please notice that the lesson for “1-place addition with sums from 11 to 18” uses the expression $9 + 5$ as its title.

Under the title “ $9 + 5$ ” is a frame broken into two lines, each with 10 blocks. In the first line, the 9 black semi-circles illustrate the first addend 9. The 5 gray semi-circles illustrate the second addend broken into two parts 1 and 4. The 1 appears on the first line, joining the black semi-circle, and filling the 10 blocks. The other 4 appears on the second line.

The corresponding expression is:

$$9 + 5 = 9 + (1 + 4) = (9 + 1) + 4 = 14$$

The illustration in the frame, accompanied by the expression explains the rationale for 1-place addition with sums of 11 to 18.

When two 1-place numbers are added, if the sum is larger than 10, an addend needs to be separated into two parts. One part joins the other addend to form a ten, which corresponds to the 1 at the tens place of the sum. The other part corresponds to the numeral at the ones place. The base-ten positional notation that we use requires this. The associative law allows this to be done.

The key skill to implement this rationale is to decide which of the two addends to separate, and how to separate. The more reasonable approach is to separate the smaller of the two addends because it is easier to “see” the quantity that composes a ten with a larger number, and thus is easier to decide how to separate the other addend.

To let students understand this rationale and acquire skill in implementing it, “the computation of adding a number to 9” serves as the best example. Because 9 is the digit closest to 10, it is easiest for students to “see” that the number needed for joining 9 to make a ten is 1. On the other hand, 1 is the number that when taken from another number has the easiest difference to determine. Therefore, in terms of intellectual load, the tasks of adding a number to 9 are the easiest of all the 1-place addition computations with sum between 11 and 18. Compare the two additions: $9 + 6$ and $7 + 5$. For the first, we need a 1 to join 9 in order to make a 10. We subtract 1 from 6 and get 5, then to combine 10 and 5 into the sum 15. Although each addend

of $9 + 5$ is larger, this computation is easier than $7 + 5$. For that, we need 3 to join 7 to make a 10. We subtract 3 from 5 and get 2, then combine 10 and 2 to get the sum 12. Now we can notice that to use “ $9 + 5$ ” as the title of the lesson is very thoughtful: with task of adding a number to 9, students get exposed to the “core technique” for 1-place additions with sums of 11 to 18 with least intellectual load. Once the computation of “ $9 + 5$ ” makes sense for students, it can serve as a template for addition computations with other numbers in this stage.

After analyzing the rationale of computing $9 + 5$, the lesson presents another calculation, $8 + 3$, using the same frame accompanied by the analogous expression to display the approach to calculation. In terms of difficulty, adding a number to 8 is the least increase from adding a number to 9. The contrast between $9 + 5$ and $8 + 3$ is a good way to help students find the pattern to use for calculating 1-place additions with composing. Adding a number to 9 is 10 plus the number minus 1, adding a number to 8 is 10 plus the number minus 2, and so on.

The four problems in the exercise section, besides requiring students to find the solution, also require them to explain the rationale. The first two problems, $9 + 7$ and $8 + 5$, are closely related to those shown earlier and can be solved with a small variation of the same approach. The other two problems, $7 + 6$ and $6 + 5$, require students to extend what they have learned to new situations. In fact, the four situations, adding a number to 9, adding a number to 8, adding a number to 7, and adding a number to 6, take care of all the situations encountered in 1-place addition with sums between 11 and 18.^{xii} The pages that follow this lesson contain more exercises involving these situations.

“1-place addition and subtraction with sums and minuends between 11 and 18” is the last stage of 1-place addition and subtraction. It appears that only two thinking tools are needed to explain the rationale of the approach: base-10 positional notation and associative law of addition. However, to use these two thinking tools to extrapolate fluently solutions at this stage requires a particular foundation. This foundation has two features:

- To master the combinations of 2 to 10 and the corresponding addition and subtraction.
- To have the habit of extrapolation and know how to use thinking tools such as commutative law, inverse operation, etc.

Fortunately, the instruction in the first three stages prepares students to have this foundation. Of the nine steps shown in Figure 3, students already have reached the seventh step before they begin the fourth stage. Only two steps are left. The instruction of extrapolation started with the capacity to mentally calculate with small numbers, that students knew before they began school. At the end of the fourth stage, students have developed the capacity to mentally calculate 1-place addition and subtraction. Moreover, they are led to build a foundation for future mathematics learning by acquiring thinking tools in nine steps.

Thinking tools vs counters or “number facts”

These examples give a brief, but comprehensive picture of how the thinking tools needed for extrapolation are introduced to first graders. Between the extrapolation approach and the counting and memorizing approaches familiar to US readers, there are some similarities and some differences.

On the one hand, the extrapolation approach is similar to the memorizing facts approach—both have a significant amount of written exercises. However, the memorizing facts approach emphasizes students’ memorizing of facts that they do not participate in developing. The extrapolation approach encourages and helps students to figure out the solutions of additions and subtractions. Students’ intellectual activities reinforce the results of calculations, and develop their capacity for mental calculation. In fact, when we say “calculating,” usually this includes thinking. The extrapolation approach intends to teach first graders how to calculate by thinking.

On the other hand, the extrapolation approach is similar to the counting approach—both draw on the computational capabilities that students bring to school and encourage students to find solutions on their own. However, the counting approach encourages students to continue using fingers or other counters, without specific attention to mental calculation. In contrast, the extrapolation approach draws on students’ primary concepts of quantities and gradually introduces mathematical thinking tools. With these thinking tools, students’ ability to mentally calculate 1-place additions and subtractions is developed, step by step. Except for small numbers such as those corresponding to perceivable quantities that children already know before school, 1-place addition and subtraction is beyond most students at the beginning of first grade. The essential difference between the extrapolation approach and the counting approach occurs when students reach the limit of their computational capacity. One approach relies on physical objects to expand students’ computation abilities and one relies on mental thinking tools to expand abilities. The extrapolation approach does not encourage students to use fingers. Some Chinese elementary teachers use the metaphor of weaning to explain that students have to give up using fingers so that they can focus on developing mental computational capability. Indeed, the instruction for extrapolation needs more carefully designed lessons and exercises to ensure that students cross the gap between the habit of counting fingers and the ability to mentally calculate 1-place additions and subtractions.

This kind of care in instructional design is illustrated by the lesson examples from the Russian textbook. Each learning task in each lesson presents only a small challenge, although its final goal is very demanding. Students are intended to acquire the capacity to calculate mentally—not only within 20, but within 100. In each learning task, the textbook shows remarkable consideration of students’ intellectual load. Every time when a new concept is introduced, the load in calculation is reduced. The textbook contains a few thousand exercises, which are connected problem by problem, and group by group. With these deliberate connections, students are led to develop their computational capacity by meeting many small challenges.

Three questions for further consideration

At the beginning of this article, I noted the importance of mental calculation of 1-place addition and subtraction in elementary mathematics learning and suggested that US elementary mathematics education might reflect on this issue. In this article, I described a kind of instruction unfamiliar to US readers—using thinking tools to extrapolate unknown from known. In conclusion, I would like to raise three questions.

Question 1: Is calculation necessarily mechanical without requiring thought?

During recent decades in US mathematics education, calculation has been viewed as unimportant. Calculation has been viewed as related to mechanical, rote learning and separate from advanced thinking, conceptual understanding, and problem solving.

Now, can what has been described in this article serve as a counterexample, showing that calculation is not necessarily mechanical, and not necessarily the product of rote learning, but can involve intellectual activity?

If calculation doesn't have to be mechanical but can be conducted as an intellectual activity, then how did this misunderstanding develop?

Question 2: How to deal with the knowledge children bring to school?

When first graders start school, they have certain mathematical capabilities. They bring some mathematical knowledge to school. In terms of 1-place addition and subtraction, their computational skills include: 1) mental calculation with small quantities such as perceivable quantities, 2) the ability to use fingers to calculate with quantities larger than those they can compute mentally. Children use both of these skills and both are equally important to them.

But, what is interesting is that the counting approach and extrapolation approach both draw on knowledge students already have, but each draws only on one kind of skill.

The counting approach draws on students' skill in computing with fingers or other counters. The task of instruction is to help students to replace their initial counting strategies with more advanced ones, so that students can find the solutions for additions and subtractions of 1-place numbers efficiently and proficiently.

The extrapolation approach draws on children's capacity for mental calculation with small quantities. The mastery of perceivable and other small quantities reflects their concepts of quantities—the magnitudes of quantities and the relationships of quantities. Based on this foundation, the extrapolation approach introduces thinking tools to children step-by-step, leading them to find the unknown from the known, using these thinking tools, gradually expanding their capability for mental calculation, and eventually developing their capability to fluently conduct mental 1-place addition and subtraction.

In terms of how to deal with the mathematical knowledge that students bring to school, what would we like to accomplish with it? The counting approach and extrapolation approach have

different goals. The counting approach aims to retain “children’s mathematics,” encouraging and helping students to solve computational problems with their “own methods.” The extrapolation approach aims to connect children to formal mathematics from the beginning: to learn to calculate with the methods of the discipline. Because their aims differ, the two approaches attend to different things, adopt different instructional methods, and achieve different results.

The two approaches also differ with respect to students’ intellectual load. The intellectual load of the counting approach is no difficulties, no accumulation. The development of the different counting strategies can occur without instruction and the transition from lower-level strategies to more advanced ones occurs without difficulty. When a more efficient strategy replaces a less efficient strategy, the previous strategy becomes useless. Thus, students do not accumulate these strategies. For example, when a student replaces counting all by counting from first, he or she will feel that computing with the new strategy is easier and more efficient. Once the old strategy of counting all is replaced, it is no longer meaningful.

The intellectual load of extrapolation approach is low difficulty with accumulation. Each step has some difficulties, but these are not so large that students can’t overcome them. Each new thinking tool is introduced sequentially, but an one is not replaced by a later one. Each continues to play a role, sometimes in cooperation with others. For example, during the stage of “addition and subtraction with sum and minuend from 6 to 9,” the commutative law, inverse operations, and compensation law are introduced. The earlier thinking tools are still useful and are used at this and later stages, and throughout elementary mathematics learning.

As they learn counting strategies, students move ahead and learn how to efficiently get solutions for 1-place additions and subtractions. However, their mental calculation capability and capacity for abstract thinking does not develop significantly. With the extrapolation approach, through learning and using the thinking tools, students not only develop mental calculation ability, but also improve their abstract thinking.

When children become first graders, how should we deal with the knowledge that they bring to school? Let them retain and fully develop “children’s mathematics” or put them on a road designed to lead them away from “children’s mathematics” to a closer connection with formal mathematics? If we let “children’s mathematics” develop fully, what will the outcome be? Can it naturally develop into formal mathematical knowledge? The computational capabilities of Brazilian child candy sellers impressed the field of mathematics education, but are those capacities the same as knowledge of the discipline of mathematics? Can it naturally transfer to more formal mathematical knowledge? And can US young people automatically develop their computational abilities to such a level? Even if they can, is that what we want?

Question 3: Is there any real “children’s mathematics”?

In present-day elementary mathematics education, the mathematical knowledge that children bring to school usually receives a significant amount of attention. The existence of “children’s mathematics” is also the theoretical foundation for the counting approach. But, what is the

mathematical knowledge that children bring to school? How long is the mathematics that children bring to school unaffected by schooling? Is there any pure “children’s arithmetic”?

The concept of “children’s mathematics” can be traced to Piaget’s research on the development of children’s mathematical capacities. In the late 1960s, Piaget’s work was welcomed by the field of mathematics education and terms such as “children’s mathematics” and “children’s arithmetic” became more frequent in mathematics education research. However, in doing so both authors and audience seem to have ignored an important fact: *except for the short period of infancy, pure children’s arithmetic does not exist.*^{xiii}

The truth is: *children’s cognitive development with respect to mathematics or any other cultural artifact occurs in an educational context—the cognitive environments created by adults.* Adults create two such environments for children’s intellectual development: informal and formal education. A child’s cognitive development is the product of the interaction between natural endowment and cognitive environments.

In mathematics, for example, before attending school, children’s cognitive environments include informal education. In everyday life, the adults who care for them conduct oral “mathematics education” spontaneously. In her work, *Children’s Counting and Concepts of Number* (1988), the mathematics education researcher Karen Fuson recorded examples of her own two daughters’ development of mathematical knowledge in a diary, and summarized other relevant research on this topic. All the examples in the book involve interactions between children and adults. Whether or not they were highly educated, all the adults strategically created cognitive environments to develop children’s concept of numbers.^{xiv} In contrast, imagine that a child has no contact with any people or a group of children has no contact with adult society. Can these children develop the “children’s mathematics” or “children’s arithmetic” that researchers have observed? The answer is likely to be no. What children bring to school is not “children’s mathematics” created only by children, but the results of interaction between natural endowment and cognitive environments that include informal education.

After entering school, children’s cognitive development acquires an additional arena—the cognitive environment of formal education. For thousands of years of human civilization, formal education was generally for the privileged few. Universal formal education, even in developed countries like the US, is only a little over a hundred years old. In general, it occurs in a special place with professional teachers, during a fixed time, and relies heavily on texts. Once formal education starts, it begins an interaction between children’s cognition and a new cognitive environment. Thus, if we say that “children’s mathematics” is the mathematical knowledge that children bring to school as a result of informal education, theoretically, this is only the situation on their first day of school. As soon as they receive formal mathematics education in school, children’s mathematical knowledge becomes the result of an interaction of their prior knowledge, the environment of school mathematics education, and their out-of-school environment. Different school mathematics education environments may have different impacts on students’ mathematical knowledge. For example, suppose two students have similar mathematical knowledge before attending school. One studies in an environment that promotes the counting approach and one studies in an environment that promotes the extrapolation

approach. Both learn 1-place addition and subtraction, but after a few weeks their mathematical knowledge and skill are likely to have obvious differences.

Figure 12 illustrates the conceptual framework just discussed (each box separated by a dotted line represents one year). Piaget’s stages of cognitive development are shown as a reference in the figure.

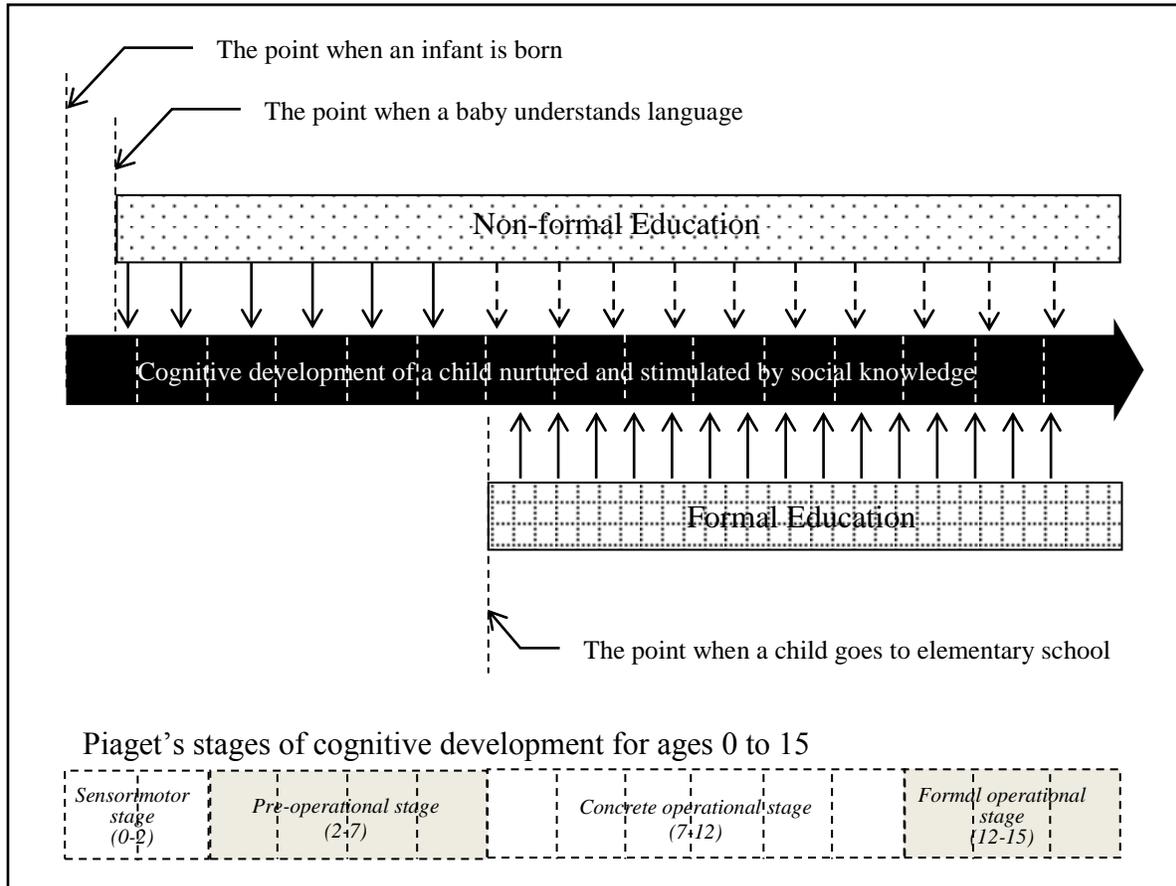


Figure 11.

Before going to school, children's cognitive environment is mainly informal education. After beginning school, formal education becomes their main cognitive environment, but informal education may still have an impact.

Piaget may have been the first researcher to have systematically studied the development of children's mathematical knowledge and to establish a theory of their development. However, his stages for children's cognitive development from birth to age 15 do not reflect the impact of an educational environment created by adults. This omission was an inevitable result of his focus on genetic epistemology.

As we know, Piaget devoted most of his career to the establishment of genetic epistemology, a field that concerns how prehistoric humans developed knowledge. In the late 1960s, toward the end of his life, Piaget summarized the theory he had established:

Genetic epistemology attempts to explain knowledge, and in particular scientific knowledge, on the basis of its history, its sociogenesis, and especially the psychological origins of the notions and operations upon which it is based. (Piaget, 1968/1970, p. 1)

The goal of genetic epistemology is to explain the origin of knowledge of prehistoric humans. However, today how can we know the origin of prehistoric humans' knowledge? Piaget had a unique idea:

The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress [that our species] made in the logical and rational organization of knowledge and the corresponding formative psychological processes [of a child]. (Piaget, 1969, p. 4)

In the late nineteenth century, the German biologist Ernst Haeckel developed the notion that the development of a human embryo repeats the evolution of the human species.^{xv} This recapitulation theory became popular in the Western world for several decades. The fundamental hypothesis of genetic epistemology was as Piaget described it—the cognitive development of individual children repeats the evolution of human knowledge, extending Haeckel's theory. With this extension, Piaget came up with an ambitious idea:

With this hypothesis, the most fruitful and the most obvious field of study [of epistemology] would be the reconstituting of human history—the history of human thinking in prehistoric man. (Piaget, 1969, p. 4)

Because the psychological development of an individual child's cognition recapitulates the corresponding historical development, then by studying children, the development of human knowledge can be reconstructed:

Unfortunately, we are not very well informed in the psychology of primitive man, but there are children all around us. It is in studying children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, and physical knowledge. (Piaget, 1969, p. 4)

After expanding recapitulation theory based on embryology, Piaget found an approach to reconstructing “the development of logical knowledge, mathematical knowledge, and physical knowledge” of prehistoric humans, by studying children. This underlies the methodological approach of genetic epistemology and was the reason why Piaget studied children’s cognitive development.

Now we can explain why in Piaget’s research on the development of children’s mathematical knowledge, such an important factor—the interaction between children and the educational environment created by adults—was ignored. For Piaget, children’s cognitive development was not the main focus, but a way to answer questions of epistemology. He wanted to reconstruct the cognitive development of prehistoric humans by studying children’s cognitive development. His attention to children’s development was determined by the goal of his research—as prehistoric human knowledge developed, there were no interactions between prehistoric humans and “adult humans.” In Piaget’s conceptual framework, the development of children’s mathematical knowledge occurs over time as measured by children’s ages, and is a result of children’s own activities and communication with other children. This matches the situation of the development of prehistoric humans that he wanted to reconstruct, but does not correspond to the situation in which present-day children’s cognitive development occurs.

To criticize genetic epistemology is not the aim of this article. What I would like to point out here is that the field of elementary mathematics education adopted Piaget’s notion of a children’s mathematics. But, in fact, children’s mathematics does not exist.

As teachers, authors of textbooks, and adult participants in elementary mathematics education, we long to know how students think about mathematics, but we also need to clearly notice that there is no general “how students think about mathematics.” “How students think about mathematics” is always the result of the interaction between a certain student at a certain time in a certain educational environment.

When the environments are different, the results of interaction may be different. The environments where children learn mathematics before going to school are mainly created by their caregivers. After beginning school, the main environment for a child to learn mathematics is generally the classroom. The environment is co-created by teaching materials (standards, textbooks, etc.) and the teacher’s instruction. Differences in family environment may result in differences in children’s mathematical knowledge before the children begin school. Differences in formal education may also result in differences in children’s mathematical knowledge.

To conclude, it is crucial that “children’s mathematics” not be considered to be the same as the mathematical knowledge that students already have (including mathematical concepts, computational skills, attitudes, and ways of thinking). Although the former does not exist, the latter is an important foundation for instruction. The label “children’s mathematics” suggests that there is a way that children think about mathematics which is independent from the impact of adults. The knowledge students already have, however, is the product of interaction between students and their previous education environment. As teachers, before we begin to teach our

students, we need to know what they know, how that knowledge was shaped, and how it is related to the knowledge that we are to teach. We also need to be clearly aware that once our instruction starts, it plays a significant role in shaping our students' mathematical knowledge. It forms the foundation for their further learning, as well as their attitudes and dispositions toward mathematics.

Concluding remarks

At the beginning of the article, I mentioned that 1-place addition and subtraction, as a part of “numerals and addition and subtraction within 20,” plays a significant role in laying down the first cornerstone of the foundation for students to learn mathematics. I also pointed out that there are different approaches to teaching it and described for US readers the extrapolation approach used in China. Interestingly, without this comparison and contrast, it appears to be hard for people involved with mathematics education in both countries to notice its characteristics. Those involved with elementary mathematics education in the US may not imagine that there are other ways to teach 1-place addition and subtraction besides “counting” and “memorizing facts.” Similarly, for Chinese elementary mathematics teachers, asking students to use their little minds is taken for granted. No one would make an effort to consider the essence of this approach and to give it a name, as I did in writing this article. Similarly, no one is likely to notice that properties such as the commutative law, which help students to “use their minds,” play the role of “thinking tools” for extrapolation.

The 1-place addition and subtraction just discussed is merely one part in the first knowledge chunk of elementary mathematics. Nevertheless, the ways it is approached in some sense represent the different philosophies of two whole systems of elementary mathematics education. The high scores in mathematics of Shanghai students during the latest Programme for International Student Assessment (PISA) called more US attention to mathematics education in China. Yet Americans may not know that the mathematics *Curriculum Standards* published by the Chinese Department of Education in 2001 were significantly influenced by the US National Council of Teachers of Mathematics *Standards*. It is my wish that readers in both countries will find this article beneficial in reflecting on mathematics education.

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ⁱ A summary of the research base for the counting approach and references to detailed syntheses are given in the appendix of *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al., 1999).

ⁱⁱ See Dehaene (1997) *The Number Sense*, pp. 126–129. Dehaene notes that associative memory frequently causes memorized multiplication facts to interfere with memorized addition facts.

ⁱⁱⁱ After comparing how 1-place addition and subtraction was taught in China and in the US, I decided to give the Chinese approach the name “extrapolation.” Chinese elementary teachers, however, may feel that this approach is not unusual, and take for granted that it is the only way to teach addition and subtraction.

^{iv}Dehaene (1997, p. 65) comments:

Georges Ifrah (1994), in his comprehensive book on the history of numerical notations, shows that in all civilizations, the first three numbers were initially denoted by repeatedly writing down the symbol for “one” as many times as necessary, exactly as Roman numerals. And most, if not all civilizations stopped using this system beyond the number 3. . . . Even our own Arabic digits, although they seem arbitrary, derive from the same principle. Our digit 1 is a single bar, and our digits 2 and 3 actually derive from two or three horizontal bars that became tied together when they were deformed by being handwritten. Only the Arabic digits 4 and beyond can thus be considered as genuinely arbitrary.

^v Between 2000 and 2005, I tested students in five different kindergartens in California. One in Palo Alto, two in East PA, one in Sacramento, one in Mountain View. Among them, the only “regular” school was in Palo Alto, students at the other four schools, more than half came from non-English speaking low-income parents without college education. Many of these children didn’t speak English and were not able to count in English. I held a random number of small plastic cubes in my fist, inviting students to play a game with me. When I opened my fist, they must immediately say how many cubes were on my palm, without counting. They didn’t need to use English, but were allowed to use their own language (Spanish or Tongan). I told them that if anyone played with me five times and got all the answers correct then he or she would win and get a picture of themselves as a prize (in fact, if they got a wrong answer, I would close my fist immediately and not let them know if they made a mistake, and continue for five rounds of the game, then take their picture). There were more than a hundred children who participated in this game. The result was that when I had 1, 2, or 3 cubes in my hand, these children never made a mistake. But when there were 4 cubes, they started to make mistakes. When the number increased to 5 or 6, there were much more errors and I could feel that they were guessing. I had encountered a situation where one child could not speak numbers but indicated the number of cubes in my hand by showing the appropriate number of fingers on his hand. When there were 2 cubes, he showed me 2 fingers; and showed me 3 fingers when I had 3 cubes in my hand.

^{vi} Concrete numbers are the numbers related to objects, for example, four apples, five candies are all concrete numbers. The numbers unconnected with particular objects, such as 4 and 5, are called abstract numbers. Smith emphasized that distinguishing between concrete numbers and abstract numbers is very important for elementary mathematics education. In fact, parents and teachers may notice that in early elementary grades, computing with concrete numbers is significantly easier for students (1925, pp. 11 – 12).

^{vii} For example, the Chinese elementary curriculum used to allocate about 20 weeks in the first semester for understanding the numerals within 20 and addition and subtraction within 20. In this period, besides developing mental calculation capability, students establish the concept of the sum of two numbers and, based on this, the concepts of addition and subtraction.

^{viii} Details of this evolution are given in Florian Cajori’s *History of Mathematical Notations*, see, e.g., signs of addition and subtraction, pp. 229–250, signs of equality, pp. 297–309. Its section on use of arithmetic and algebraic notation by individual writers (pp. 71–229) illustrates the long evolution of arithmetic and algebraic syntax.

^x In fact, the application of compensation law of multiplication to division is the key idea underlying the algorithms for the four operations with fractions.

^{xi} The associative law of addition expressed in words is “A sum doesn’t depend on grouping of its addends,” expressed in letters, it is $a + b + c = (a + b) + c = a + (b + c) = (a + c) + b$. The rationale of addition with composing is when two 1-place numbers are added, if the sum is larger than 10, to express one of the addends as the sum of two addends, one of them should be able to make a ten with the first addend. Make a ten first, then add to the remainder of the addend. *If $a + b > 10$, then $a + b = a + (c + d) = (a + c) + d$; ($b = c + d$; $a + c = 10$).*

^{xii} Calculations that involve adding a number to 9 are not only easiest, but are the most frequent in the sums of the sums of this stage. Addition calculations with sums of 11 to 18 are based on 20 different pairs of numbers. Among these 20 pairs, eight of them, two-fifths of all the pairs, include 9. Six of the remaining pairs, three-tenths of the total, contain 8; four of the remaining pairs contain 7 and the last two pairs contain 6. The proportion is 4:3:2:1.

^{xiii} Some research on infants’ mathematics suggests that it is independent from adults’ impact. For example, the research of Harvard University professor Karen Wynn finds that babies as young as four months “knew” $1 + 1 = 2$, that $1 + 1 = 1$ and $1 + 1 = 3$ are both false, and that $1 + 2 = 3$. These findings, if they are robust (some researchers question this work), would imply that human babies are born knowing the quantities 1, 2, 3 and “ $1 + 1 = 2$,” etc.

^{xiv} In summarizing research on this topic, Fuson noted three features:

First, “Mothers seem to use number words more with very young children and then decrease their use as the children begin to use number words more frequently.”

Second, mothers “did not always just use the number word, but rather directed the child’s attention to the appropriate attributes of a situation.

Third, after such an initial structuring, the mother might then continue to restructure the same situation into other uses in a sequential goal-directed fashion. . . . Mothers made judgments about what steps their child could carry out and then structured the steps into greater or lesser difficulty accordingly (Fuson, 1988, p. 16).

^{xv} Haeckel wrote a popular book, which supported his theory with scientific drawings showing human and other embryos passing through various stages of development. Questions about the accuracy of these drawings were raised soon after its publication in 1868. The theory generated interest among those concerned with social and educational issues in the late nineteenth century. The specific form given by Haeckel, that the embryo as it develops *repeats* the prior evolutionary stages of its species, slowly became discredited among biologists.